KSU CET UNIT

FIRST YEAR NOTES

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* Representation of f(x) by a Fourier integral

Theorem 1 If $f(x)$ is real valued, absolutely integrable ($\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$) and piecewise smooth, then the Fourier integral of $f(x)$ is defined by

$$\int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega \quad \text{(1)}$$

where $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx$$

Remark: 1) The Fourier integral of $f(x)$ is $f(x)$ at all points at which $f(x)$ is continuous.

2) The F.I of $f(x) = \frac{f(a+)+f(a-)}{2}$ if $f(x)$ is discontinuous at $x=a$.

Qn 1) Find the Fourier integral representation of the function $f(x) = \begin{cases} 1 & \text{for } |x|<1 \\ 0 & \text{otherwise} \end{cases}$

Hence evaluate $\int_0^{\infty} \frac{\sin\omega}{\omega}\,d\omega$

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Soln

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\, dx$$

$$= \frac{1}{\pi}\int_{-1}^{1}\cos\omega x\, dx \quad \left(= \frac{2}{\pi}\int_{0}^{1}\cos\omega x\, dx\right)$$

$$= \frac{1}{\pi}\left[\frac{\sin\omega x}{\omega}\right]_{-1}^{1} = \frac{2\sin\omega}{\pi\omega}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\, dx$$

$$= \frac{1}{\pi}\int_{-1}^{1}\sin\omega x\, dx = 0 \text{ (odd fn)}$$

$$f(x) = \int_{0}^{\infty}\frac{2\sin\omega}{\pi\omega}\cos\omega x\, d\omega$$

$$= \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin\omega}{\omega}\cos\omega x\, d\omega \quad \text{—(1)}$$

At $x = \pm 1$, the above integral is

$$\frac{f(1+)+f(1-)}{2} = \frac{0+1}{2} = \frac{1}{2}$$

$$\therefore \text{FI of } f(x) = \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin\omega}{\omega}\cos\omega x\, d\omega$$

$$= \begin{cases} 1 & \text{for } |x|<1 \\ 0 & \text{for } |x|>1 \\ 1/2 & \text{for } x = \pm 1 \end{cases}$$

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When $x=0$ we have

$$\int_0^\infty \frac{\sin\omega}{\omega}\, d\omega = \frac{\pi}{2}.$$

★ **Fourier Cosine integral**

If $f(x)$ is a real function on $0<x<\infty$, satisfying the conditions in Theorem-1, then the Fourier Cosine integral of $f(x)$ is defined by

$$\int_0^\infty A(\omega)\cos\omega x\, d\omega$$

where $A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\, dx.$

(ie, if $f(x)$ has a Fourier integral representation and is even then $B(\omega)=0$ and we have the Fourier cosine integral.)

Remark 1. The Fourier Cosine integral of $f(x)$ is $f(x)$ for all points at which $f(x)$ is continuous

2. FCI of $f(x) = \frac{f(a+) + f(a-)}{2}$, if $f(x)$ is discontinuous at $x=a$.

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Qn 2) Find the Fourier Cosine integral of

$$f(x) = \begin{cases} e^{-kx}, & x \geq 0 \\ 0 & \text{otherwise} \end{cases}, \quad k > 0.$$

Hence evaluate $\int_0^\infty \frac{\cos\omega x}{k^2+\omega^2}\, d\omega$.

Soln.

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\, dx$$

$$= \frac{2}{\pi}\int_0^\infty e^{-kx}\cos\omega x\, dx$$

$$= \frac{2}{\pi}\left[\frac{e^{-kx}}{k^2+\omega^2}\left(-k\cos\omega x + \omega\sin\omega x\right)\right]_0^\infty$$

$$= \frac{2}{\pi}\left[0 - \frac{(-k)}{k^2+\omega^2}\right]$$

$$= \frac{2k}{\pi(k^2+\omega^2)}$$

$\therefore$ Fourier Cosine integral is

$$f(x) = \int_0^\infty \frac{2k}{\pi}\frac{\cos\omega x}{k^2+\omega^2}\, d\omega$$

ie, $e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos\omega x}{k^2+\omega^2}\, d\omega, \quad x > 0, \; k >$

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Hence
$$\int_0^\infty \frac{\cos \omega x}{k^2+\omega^2}\, d\omega = \frac{\pi}{2k} e^{-kx}, \quad x>0, k>0$$

* Fourier Sine Integral

If $f(x)$ is a real function on $0<x<\infty$, satisfying the conditions in Theorem-1, then the Fourier Sine integral of $f(x)$ is defined by
$$\int_0^\infty B(\omega) \sin \omega x \, d\omega$$
where
$$B(\omega) = \frac{2}{\pi} \int_0^\infty f(x) \sin \omega x \, dx.$$

(ie, if $f(x)$ has a Fourier integral repsn and is odd then $A(\omega)=0$ and we have the Fourier Sine integral)

Remark 1. Fourier Sine integral of $f(x)$ is $f(x)$ for all points at which $f(x)$ is continuous

2. FSI of $f(x) = \dfrac{f(a+)+f(a-)}{2}$, if $f(x)$ is discontinuous at $x=a$.

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Qn 3 Find the Fourier sine integral of $f(x) = \begin{cases} e^{-kx}, & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$ $k > 0$

Hence evaluate $\int_0^\infty \frac{\omega \sin(\omega x)}{k^2 + \omega^2} d\omega$

Soln :

$$B(\omega) = \frac{2}{\pi} \int_0^\infty f(x) \sin \omega x \, dx$$

$$= \frac{2}{\pi} \int_0^\infty e^{-kx} \sin \omega x \, dx$$

$$= \frac{2}{\pi} \left[\frac{e^{-kx}}{k^2+\omega^2} (-k \sin \omega x - \omega \cos \omega x) \right]_0^\infty$$

$$= \frac{2}{\pi} \left[0 + \frac{\omega}{k^2+\omega^2} \right] = \frac{2\omega}{\pi(k^2+\omega^2)}$$

$\therefore$ FSI is $f(x) = e^{-kx} = \frac{2}{\pi} \int_0^\infty \frac{\omega \sin \omega x}{(k^2+\omega^2)} d\omega$

$x > 0, \ k > 0$

Hence $\int_0^\infty \frac{\omega \sin \omega x}{k^2+\omega^2} d\omega = \frac{\pi e^{-kx}}{2}, \ x > 0, k > 0$

The 2 integrals (previous problem) are called Laplace integrals

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Fourier Cosine Transform

If $f(x)$ is a real function on $0 < x < \infty$, satisfying the conditions in Theorem 1, then the Fourier Cosine Transform is defined by

$$\mathcal{F}_c(f(x)) = \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\omega x)\, dx$$

The inverse Fourier Cosine transform is defined by

$$\mathcal{F}_c^{-1}(\hat{f}_c(\omega)) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\omega) \cos \omega x \, d\omega$$

Remark 1. $\mathcal{F}_c^{-1}(\hat{f}_c(\omega)) = f(x)$ for all points at which $f(x)$ is continuous

2. $\mathcal{F}_c^{-1}(\hat{f}_c(\omega)) = \dfrac{f(a+) + f(a-)}{2}$ if $f(x)$ is discontinuous at $x = a$.

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* Fourier Sine Transform

If $f(x)$ is a real function on $0 < x < \infty$, satisfying the conditions in Theorem-1, then Fourier Sine transform is defined by

$$\mathcal{F}_S(f(x)) = \hat{f}_S(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\omega x)\, dx$$

The inverse Fourier Sine transform is defined by

$$\mathcal{F}_S^{-1}(\hat{f}_S(\omega)) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_S(\omega) \sin(\omega x)\, d\omega$$

Remark

1. $\mathcal{F}_S^{-1}(\hat{f}_S(\omega)) = f(x)$ for all points at which $f(x)$ is continuous.

2. $\mathcal{F}_S^{-1}(\hat{f}_S(\omega)) = \dfrac{f(a+) + f(a-)}{2}$ if $f(x)$ is discontinuous at $x = a$.

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Qn 4) Find the Fourier Cosine and Sine transform of

$$f(x) = \begin{cases} k & 0<x<a \\ 0 & x>a \end{cases}$$

Soln.

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\omega x)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a k \cos(\omega x)\, dx$$

$$= \sqrt{\frac{2}{\pi}}\, k \left(\frac{\sin \omega x}{\omega}\right)_0^a$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{k \sin a\omega}{\omega}$$

$|||^{ly}$ $$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a k \sin \omega x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\, k \left(-\frac{\cos \omega x}{\omega}\right)_0^a$$

$$= \sqrt{\frac{2}{\pi}}\, k \frac{(1-\cos a\omega)}{\omega}$$

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Qn) 5 Find the Fourier Cosine and Sine transforms of e^{-kx}, $k>0$.

<u>Soln</u>

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\omega x)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-kx} \cos(\omega x)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-kx}}{k^2+\omega^2} (-k\cos\omega x + \omega \sin\omega x) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{k}{k^2+\omega^2}$$

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\omega x)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-kx} \sin(\omega x)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-kx}}{k^2+\omega^2} (-k\sin\omega x - \omega \cos\omega x) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{\omega}{k^2+\omega^2}$$

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Qn 6) Find the Fourier Cosine transform of e^{-kx^2}, $k>0$.

Soln :

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\omega x)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-kx^2} \cos(\omega x)\, dx$$

$$= I \quad \text{—— (1)}$$

Then

$$\frac{dI}{d\omega} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-kx^2} \times -\sin(\omega x) \times x\, dx$$

$$= \frac{1}{2k} \sqrt{\frac{2}{\pi}} \int_0^\infty (\sin \omega x)(-2kx e^{-kx^2} dx)$$

$$= \frac{1}{2k} \sqrt{\frac{2}{\pi}} \left[\left(\sin \omega x \cdot e^{-kx^2}\right)_0^\infty - \int_0^\infty \omega \cos \omega x \times e^{-kx^2} dx \right]$$

$$= \frac{-\omega}{2k} \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-kx^2} \cos(\omega x)\, dx$$

$$= -\frac{\omega}{2k} I \quad \text{From (1)}$$

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$$\therefore \frac{dI}{I} = \frac{-\omega d\omega}{2k}$$

On integrating

$$\int \frac{dI}{I} = -\frac{1}{2k} \int \omega d\omega$$

$$\log I = -\frac{\omega^2}{4k} + \log A$$

$$\therefore I = A e^{-\omega^2/4k} \quad \text{--- (2)}$$

When $\omega = 0$, from (1) we have

$$I = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-kx^2} dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sqrt{\pi}}{2\sqrt{k}} = \frac{1}{\sqrt{2k}}$$

From (2) $I = A$

$$\therefore A = \frac{1}{\sqrt{2k}}$$

$$\therefore F_c(f(x)) = \frac{1}{\sqrt{2k}} e^{-\frac{\omega^2}{4k}}$$

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Properties of Fourier cosine and sine transforms

1. **Linearity**

i) $\mathbb{F}_C\left[a f(x) + b g(x)\right] = a \mathbb{F}_C(f(x)) + b \mathbb{F}_C(f(x))$

ii) $\mathbb{F}_S\left[a f(x) + b g(x)\right] = a \mathbb{F}_S(f(x)) + b \mathbb{F}(g(x))$

where a and b are constants

2. **Cosine and Sine transform of derivatives**

If $f(x)$ is continuous, absolutely integrable and $f(x) \to 0$ as $x \to \infty$, then

i) $\mathbb{F}_C(f'(x)) = \omega \mathbb{F}_S(f(x)) - \sqrt{\frac{2}{\pi}} f(0)$

ii) $\mathbb{F}_S(f'(x)) = -\omega \mathbb{F}_C(f(x))$

iii) $\mathbb{F}_C(f''(x)) = -\omega^2 \mathbb{F}_C(f(x)) - \sqrt{\frac{2}{\pi}} f'(0)$

iv) $\mathbb{F}_S(f''(x)) = -\omega^2 \mathbb{F}_S(f(x)) + \sqrt{\frac{2}{\pi}} \omega f(0)$

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ms Qn 7) Find the Fourier cosine and sine transforms of e^{-kx}, $k>0$, by using the above property.

Soln:

Let $f(x) = e^{-kx}$

$\therefore f'(x) = -ke^{-kx}$

& $f''(x) = k^2 e^{-kx}$

$$\text{ie, } k^2 f(x) = f''(x) \quad \text{——} \ ①$$

$$F_C\left(k^2 f(x)\right) = F_C\left(f''(x)\right)$$

$$\Rightarrow \quad k^2 F_C\left(f(x)\right) = -\omega^2 F_C\left(f(x)\right) - \sqrt{\frac{2}{\pi}}\, f'(0)$$

$$= -\omega^2 F_C\left(f(x)\right) + k\sqrt{\frac{2}{\pi}}$$

$$\Rightarrow \quad (k^2+\omega^2) F_C\left(f(x)\right) = k\sqrt{\frac{2}{\pi}}$$

$$\therefore F_C\left(f(x)\right) = \sqrt{\frac{2}{\pi}}\ \frac{k}{k^2+\omega^2}$$

From ①

$$k^2 F_S\left(f(x)\right) = F_S\left(f''(x)\right)$$

$$= -\omega^2 F_S\left(f(x)\right) + \sqrt{\frac{2}{\pi}}\ \omega \times 1$$

$$\therefore F_S\left(f(x)\right) = \sqrt{\frac{2}{\pi}}\ \frac{\omega}{k^2+\omega^2}$$

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Qn 8) Find the Fourier sine transform of $xe^{-x^2/2}$

Soln:

Let $f(x) = e^{-x^2/2}$

$\therefore f'(x) = -x e^{-x^2/2}$

$-f'(x) = x f(x)$

Applying FST

$F_S(xf(x)) = F_S(-f'(x))$

$\Rightarrow F_S(xe^{-x^2/2}) = -F_S(f'(x))$

$= -(-\omega F_C(e^{-x^2/2}))$

$= \omega F_C(e^{-x^2/2}) = \omega e^{-\omega^2/2}$

$\left[\text{From Qn 6, } F_C(e^{-x^2/2}) = \frac{1}{\sqrt{2\times\frac{1}{2}}} e^{-\frac{\omega^2}{2}} = e^{-\omega^2/2}\right]$

Qn Find the cosine transform of $f(x) = \begin{cases} \cos x, & 0<x<1 \\ 0 & x>1 \end{cases}$

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x \, dx$$

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$$= \sqrt{\frac{2}{\pi}} \int_0^1 \cos x \cos(\omega x)\, dx .$$

$$= \sqrt{\frac{2}{\pi}} \times \frac{1}{2} \int_0^1 2\cos x \cos(\omega x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^1 \left[\cos(x+\omega x) + \cos(x-\omega x)\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^1 \left[\cos(1+\omega)x + \cos(1-\omega)x\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(1+\omega)x}{(1+\omega)} + \frac{\sin(1-\omega)x}{(1-\omega)}\right]_0^1$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(1+\omega)}{(1+\omega)} + \frac{\sin(1-\omega)}{(1-\omega)}\right]$$

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Fourier Transform

If $f(x)$ is real valued, absolutely integrable and piecewise smooth, then the Fourier transform of $f(x)$ is defined by

$$\mathcal{F}(f(x)) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

The inverse Fourier transform is defined by

$$\mathcal{F}^{-1}(\hat{f}(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$$

Remark

1. $\mathcal{F}^{-1}(\hat{f}(\omega)) = f(x)$ for all points at which $f(x)$ is continuous

2. $\mathcal{F}^{-1}(\hat{f}(\omega)) = \dfrac{f(a+) + f(a-)}{2}$, if $f(x)$ is discontinuous at $x = a$.

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Qn 9. Find the Fourier transform of $f(x) = \begin{cases} 1 & \text{for } |x|<1 \\ 0 & \text{otherwise} \end{cases}$

Hence evaluate $\int_{-\infty}^{\infty} \frac{\sin\omega}{\omega}\, d\omega$

Soln:

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (\cos\omega x - i\sin\omega x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \times 2\int_{0}^{1} \cos\omega x\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin\omega x}{\omega}\right]_0^1$$

$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}}\, \frac{\sin\omega}{\omega}$$

$$F^{-1}(\hat{f}(\omega)) = \frac{1}{\sqrt{2\pi}} \times \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{\sin\omega}{\omega} e^{i\omega x} d\omega$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin\omega}{\omega} (\cos\omega x + i\sin\omega x)\, d\omega$$

$$= \begin{cases} 1 & |x|<1 \\ 0 & |x|>1 \\ \frac{1}{2} & x = \pm 1 \end{cases}$$

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When $x=0$. we have

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\omega}{\omega}\, d\omega = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin\omega}{\omega}\, d\omega = \pi$$

Qn 10. Find the Fourier transform of $f(x) = \begin{cases} e^{-kx}, & x> \\ 0 & \text{otherwise} \end{cases}$

$k>0$

Soln :

$$F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-kx}\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(k+i\omega)x}}{-(k+i\omega)} \right]_0^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{k+i\omega} \right)$$

Qn

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Qn 11. Find the Fourier transform of e^{-kx^2}, $k>0$

Soln.

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-kx^2} e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(kx^2 + i\omega x + \left(\frac{i\omega}{2\sqrt{k}}\right)^2 - \left(\frac{i\omega}{2\sqrt{k}}\right)^2\right)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\sqrt{k}x + \frac{i\omega}{2\sqrt{k}}\right)^2} e^{\left(\frac{i\omega}{2\sqrt{k}}\right)^2} dx$$

$$= \frac{e^{-\frac{\omega^2}{4k}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\sqrt{k}x + \frac{i\omega}{2\sqrt{k}}\right)^2} dx$$

put $\sqrt{k}x + \frac{i\omega}{2\sqrt{k}} = u$

$\Rightarrow \sqrt{k}\,dx = du$

$$= \frac{e^{-\frac{\omega^2}{4k}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{\sqrt{k}}$$

$$= \frac{e^{-\frac{\omega^2}{4k}}}{\sqrt{2\pi k}} \int_{-\infty}^{\infty} e^{-u^2} du \quad \text{——— (1)}$$

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Let $I = \int_{-\infty}^{\infty} e^{-x^2} dx$

$$\therefore I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\,dy$$

Changing to polar coordinates, $x = r\cos\theta, y = r\sin\theta$

$x^2+y^2 = r^2$ & $dx\,dy = r\,dr\,d\theta$

Also θ varies from 0 to 2π and

r varies from 0 to ∞

$$\therefore I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2} r\,dr\,\cancel{d\theta} \qquad r^2 = t$$

$$2r\,dr = dt$$

$$= (\theta)_0^{2\pi} \int_0^{\infty} e^{-t} \frac{dt}{2}$$

$$= 2\pi \times \frac{1}{2}\left(\frac{e^{-t}}{-1}\right)_0^{1} = 2\pi \times \frac{1}{2} = \pi$$

ie, $I = \sqrt{\pi}$

From eqn (1)

$$F(f(x)) = \frac{e^{\frac{-\omega^2}{4k}}}{\sqrt{2k\pi}} \times \sqrt{\pi} = \frac{e^{\frac{-\omega^2}{4k}}}{\sqrt{2k}}$$

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Properties of Fourier transforms

1. **Linearity:**

$$\mathcal{F}(af(x)+bg(x)) = a\mathcal{F}(f(x)) + b\mathcal{F}(g(x)),$$

where a and b are constants.

2. **Shifting**

$$\mathcal{F}(f(x-a)) = e^{-i\omega a}\,\mathcal{F}(f(x))$$

3. If $f(x)$ is even in $-\infty < x < \infty$, then

$$\mathcal{F}(f(x)) = \mathcal{F}_c(f(x))$$

4. If $f(x)$ is odd in $-\infty < x < \infty$, then

$$\mathcal{F}(f(x)) = -i\,\mathcal{F}_s(f(x))$$

5. **Fourier transforms of derivatives.**

If $f(x)$ is continuous, $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$ and $f'(x)$ is absolutely integrable, then

i) $\mathcal{F}(f'(x)) = i\omega\,\mathcal{F}[f(x)]$

ii) $\mathcal{F}(f''(x)) = -\omega\,\mathcal{F}(f(x))$

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Qn 12. Find the Fourier transform of xe^{-x^2}

<u>Soln</u>: Let $f(x) = e^{-x^2}$

$$f'(x) = -2x\, e^{-x^2} = -2x f(x)$$

$$\therefore \quad x f(x) = -\frac{1}{2} f'(x)$$

Applying Fourier Transform

$$\mathcal{F}(x f(x)) = \mathcal{F}\left(-\frac{1}{2} f'(x)\right)$$

$$\Rightarrow \mathcal{F}(xe^{-x^2}) = -\frac{1}{2}\mathcal{F}(f'(x))$$

$$= -\frac{1}{2} \times i\omega\, \mathcal{F}(f(x))$$

$$= -\frac{i\omega}{2}\mathcal{F}(e^{-x^2})$$

$$= -\frac{i\omega}{2} \times \frac{e^{-\frac{\omega^2}{4}}}{\sqrt{2}} \quad \left[\begin{matrix}\text{From Qn 11} \\ \text{put } k=1\end{matrix}\right]$$

$$= \frac{-i\omega}{2\sqrt{2}}\, e^{-\frac{\omega^2}{4}}$$

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Problems

Qn. Find the FT of $f(x) = \begin{cases} |x| & \text{if } -1<x<1 \\ 0 & \text{otherwise} \end{cases}$

Soln.

Since $f(x)$ is an even function, its ~~the~~ Fourier transform is same as its Fourier Cosine transform

$$\mathcal{F}_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 x \cos(\omega x) \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[x \frac{\sin \omega x}{\omega} + \frac{\cos \omega x}{\omega^2} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \omega}{\omega} + \frac{\cos \omega}{\omega^2} - \frac{1}{\omega^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\omega \sin \omega + \cos \omega - 1}{\omega^2} \right]$$

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Qn. Find the Fourier transform of $f(x) = \begin{cases} -1, & \text{if } -1<x<0 \\ 1, & \text{if } 0<x<1 \\ 0, & \text{otherwise} \end{cases}$

Soln:

$$\mathbb{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{0} (-1)\, e^{-i\omega x}\, dx + \int_{0}^{1} e^{-i\omega x}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\int_{-1}^{0} (\cos\omega x - i\sin\omega x)\, dx + \int_{0}^{1} (\cos\omega x - i\sin\omega x)\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\int_{-1}^{0} \cos\omega x\, dx + i\int_{-1}^{0} \sin\omega x\, dx + \int_{0}^{1} \cos\omega x\, dx - i\int_{0}^{1} \sin\omega x\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\left(\frac{\sin\omega x}{\omega}\right)_{-1}^{0} + i\left(\frac{-\cos\omega x}{\omega}\right)_{-1}^{0} + \left(\frac{\sin\omega x}{\omega}\right)_{0}^{1} - i\left(\frac{-\cos\omega x}{\omega}\right)_{0}^{1} \right]$$

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$$= \frac{1}{\sqrt{2\pi}}\left[-\frac{\sin\omega}{\omega} - i\left(\frac{1-\cos\omega}{\omega}\right) + \frac{\sin\omega}{\omega}\right.$$

$$\left. i\left(\frac{\cos\omega - 1}{\omega}\right)\right]$$

$$= \frac{2i}{\sqrt{2\pi}}\left(\frac{\cos\omega - 1}{\omega}\right) = i\sqrt{\frac{2}{\pi}}\left(\frac{\cos\omega - 1}{\omega}\right)$$

2n.

Problems

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Q11) Find the complex FT and Fourier integral repsn of $f(x) = \begin{cases} e^{-kx} & \text{for } x \geq 0,\ k>0 \\ 0 & \text{otherwise} \end{cases}$

Soln:

FT, $\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{k+i\omega}\right) = \hat{f}(\omega)$ (Q11 10)

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\omega x)\,dx = \frac{k}{\pi(k^2+\omega^2)}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\omega x)\,dx = \frac{\omega}{\pi(k^2+\omega^2)}$$

$\therefore$ FI representation is

$$\int_0^{\infty}\left[A(\omega)\cos\omega x + B(\omega)\sin\omega x\right]d\omega$$

$$= \frac{1}{\pi}\int_0^{\infty}\left[\frac{k\cos(\omega x) + \omega\sin(\omega x)}{(k^2+\omega^2)}\right]d\omega$$

* FI repsn is also given by:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)\,e^{i\omega x}\,d\omega$$

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$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left[\frac{1}{k+i\omega}\right] e^{i\omega x} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{(k+i\omega)} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(k-i\omega)(\cos\omega x + i\sin\omega x)}{(k+i\omega)(k-i\omega)} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{k\cos(\omega x) + \omega\sin(\omega x)}{(k^2+\omega^2)} \, d\omega$$

$$+ \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{k\sin(\omega x) - \omega\cos(\omega x)}{(k^2+\omega^2)} \, d\omega$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \frac{k\cos(\omega x) + \omega\sin(\omega x)}{(k^2+\omega^2)} \, d\omega$$

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Convolution

The convolution of functions $f(x)$ and $g(x)$ is defined by

$$(f*g)(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du = \int_{-\infty}^{\infty} g(u) f(x-u) du$$

Convolution Theorem

Suppose that two functions $f(x)$ and $g(x)$ are piecewise continuous and absolutely integrable, then

$$\mathcal{F}(f*g) = \sqrt{2\pi}\, \mathcal{F}(f)\, \mathcal{F}(g)$$

Qn 14. Verify the convolution theorem for $f(x)=g(x)=e^{-x^2}$

Soln :

$$f(u)\, g(x-u) = e^{-u^2} e^{-(x-u)^2}$$

$$= e^{-(u^2+x^2-2xu+u^2)}$$

$$= e^{-(2u^2-2xu+x^2)}$$

$$= e^{-2\left(u^2-xu+\frac{x^2}{2}\right)}$$

$$= e^{-2\left[\left(u-\frac{x}{2}\right)^2+\frac{x^2}{4}\right]}$$

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$$= e^{-2\left(u-\frac{x}{2}\right)^2} e^{-\frac{x^2}{2}}$$

$$\therefore f*g = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

$$= \int_{-\infty}^{\infty} e^{-2\left(u-\frac{x}{2}\right)^2} e^{-\frac{x^2}{2}} du$$

$$= e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-2\left(u-\frac{x}{2}\right)^2} du$$

Put $t = \sqrt{2}\left(u-\frac{x}{2}\right) \Rightarrow dt = \sqrt{2}\, du$

$$\therefore f*g = e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{\sqrt{2}}$$

$$= \frac{e^{-x^2/2}}{\sqrt{2}} \times \sqrt{\pi} = \sqrt{\frac{\pi}{2}}\, e^{-x^2/2}$$

$$\therefore \mathcal{F}(f*g) = \mathcal{F}\left(\sqrt{\frac{\pi}{2}}\, e^{-x^2/2}\right)$$

$$= \sqrt{\frac{\pi}{2}}\, \mathcal{F}\left(e^{-x^2/2}\right)$$

$$= \sqrt{\frac{\pi}{2}}\, e^{-\omega^2/2} \quad \text{--- (1)}$$

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$$\mathcal{F}(f)\,\mathcal{F}(g) = \mathcal{F}(e^{-x^2})\,\mathcal{F}(e^{-x^2})$$

$$= \frac{1}{\sqrt{2}} e^{-\omega^2/4} \times \frac{1}{\sqrt{2}} e^{-\omega^2/4}$$

$$= \frac{1}{2} e^{-\omega^2/2}$$

$$\therefore \sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g) = \sqrt{2\pi} \times \frac{1}{2} e^{-\omega^2/2}$$

$$= \sqrt{\frac{\pi}{2}}\, e^{-\omega^2/2} \quad \text{--- (2)}$$

From (1) & (2) we have

$$\mathcal{F}(f * g) = \sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g)$$

Hence convolution theorem is verified

Qn. Find the Fourier transform of $e^{-|x|}$, $-\infty < x < \infty$.

<u>Soln</u>:

Since $f(x)$ is an even function

$$\mathcal{F}(f(x)) = \mathcal{F}_c(f(x))$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-|x|} \cos \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+\omega^2} \left(-\cos\omega x + \omega \sin \omega x \right) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{0 - (-1)}{1+\omega^2} \right] = \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2}$$